CHARACTERIZATION OF OPTIMAL LES IN TURBULENT CHANNEL FLOW

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Abstract. The application of large eddy simulation (LES) to wall-bounded turbulent flows has be hindered by the failure of current subgrid models in the strongly inhomogeneous region very near the wall. This is the LES wall modeling problem. To address this a new modeling approach called optimal LES modeling is applied to a turbulent channel flow. Ideal LES is an LES evolution that is guaranteed to produce correct statistics and accurate short-time dynamics, and optimal LES is a minimum error approximation to it. By constructing optimal models that produce correct *a priori* estimates of important statistical quantities, it is shown that for inhomogeneous flows, the subgrid model must represent Reynolds stress spatial transport, in addition to transfer to small scales. Resulting models are found to perform particularly well in LES.

1. Introduction

Large eddy simulation (LES) is a promising simulation technique in which only the large scales of a turbulent flow are simulated and the effects of the small scales are modeled. A variety of subgrid models have been developed, and using these models, LES has been successfully applied in a variety of flows (see Lesieur & Métais, 1996; and Meneveau and Katz, 2000, for reviews). Unfortunately, the near-wall region of a wall-bounded turbulent flow causes difficulties for LES, primarily due to the strong inhomogeneity of the turbulence in this region, which results in a violation of the subgrid homogeneity and isotropy on which most models are predicated. In essence, there is "large-scale" turbulence that is actually smaller than the filter scale.

A new approach to large eddy simulation model formulation (optimal LES) has been developed (Langford and Moser, 1999), which does not rely on subgrid homogeneity or isotropy. Optimal LES is the formal approximation of what we call the "ideal LES," which is the best possible deterministic LES evolution. In this paper, the optimal LES technique is applied to the turbulent flow in a channel, to begin to address the problem of near-wall LES modeling. In the following subsections the optimal LES approach will be briefly described.

1.1. FILTERING AND IDEAL LES

The large scales to be simulated in an LES are defined through a spatial filter denoted $\tilde{\cdot}$. For the filter to be useful in the LES context, it cannot be invertible (Langford and Moser, 1999). That is, it must discard information, so that the (formally infinite-dimensional) space of Navier-Stokes solutions will be mapped to a smaller-dimensional space that can be practically represented on a computer. If an invertible filter were used, the dynamics of the filtered system would be identical to the dynamics of the unfiltered system; only the variables describing it would be different. Often the explicit filter used in an LES is invertible (e.g. Gaussian or top-hat), but the numerical discretization (e.g. Fourier truncation or point sampling) invariably introduces non-invertability. In this case we include the discretization as part of the filter.

With an uninvertible filter, the LES state information (the large-scale field) is insufficient to determine either the unfiltered field or the evolution of the filtered field; thus the need for a model. There are in general many possible evolutions of a given filtered field, depending which of an infinite number of subgrid fields is present. In the absence of subgrid information, the large-scale evolution can be considered to be stochastic, and an intuitively reasonable LES evolution would be the average of all the possible large-scale evolutions. This is written mathematically as the conditional average:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \left\langle \widetilde{\frac{\mathrm{d}u}{\mathrm{d}t}} \middle| \tilde{u} = w \right\rangle, \tag{1}$$

where the LES field is w, and u is a real turbulent field. It has been shown (Langford and Moser, 1999) that this is the unique LES evolution that guarantees accurate one-time statistics *and* minimizes error of the large-scale dynamics. Because this is all one could wish for in an LES, this evolution is called *ideal* LES.

It is customary to write the LES equations as the Navier–Stokes operators operating on the filtered field:

$$\frac{\partial \tilde{u}_i}{\partial t} = -\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} - \frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j} + M_i, \tag{2}$$

where

$$M_i = -\frac{\partial \tau_{ij}}{\partial x_j} + C_i, \qquad \tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j, \tag{3}$$

and C_i is a term that arises when the filter does not commute with spatial differentiation. The ideal subgrid model ($\mathbf{m}$) for the the subgrid term ($\mathbf{M}$) is then written:

$$\mathbf{m}(w) = \left\langle \mathbf{M}(u) \middle| \tilde{u} = w \right\rangle, \tag{4}$$

1.2. OPTIMAL LES

The ideal LES defined above has all the features one could wish for in an LES model, but unfortunately it cannot be determined directly. The statistical information embodied in the conditional average of the ideal LES is so tremendous that it is unlikely that the ideal model for a given flow or filter could ever be found exactly. Still, one can approximate the ideal model, and indeed, subgrid modeling can be considered to be the problem of approximating the conditional average that defines the ideal model. The term *optimal LES* is used to describe formulations that most closely approximate ideal LES within some class. Optimal models are defined using stochastic estimation, as originally proposed by Adrian (1990). Stochastic estimation is a well-established method for approximating conditional averages (Adrian, 1977; Adrian and Moin, 1988; Adrian et al., 1989).

In stochastic estimation, a vector of random fields Y is approximated linearly in terms of a vector of "event" fields $E(\mathbf{x})$. The stochastic estimate is then written:

$$y_i(\mathbf{x}) = \langle Y_i \rangle + \int_{\mathcal{D}} L_{ij}(\mathbf{x}, \mathbf{x}') E'_j(\mathbf{x}') \mathrm{d}\mathbf{x}', \tag{5}$$

where $\langle \cdot \rangle$ is the average and $Y' = Y - \langle Y \rangle$.

The estimation kernel L_{ij} is determined from the two-point cross correlations of $E'(\mathbf{x})$ and Y' from:

$$\langle E'_i(\mathbf{x}') Y'_j(\mathbf{x}) \rangle = \int_{\mathcal{D}} L_{jk}(\mathbf{x}, \mathbf{x}'') \langle E'_i(\mathbf{x}') E'_k(\mathbf{x}'') \rangle \mathrm{d}\mathbf{x}'', \tag{6}$$

for all i, j and $\mathbf{x}'$. Nonlinear estimates are obtained by including nonlinear quantities as part of of the event vector.

In what follows, optimal LES models are devised for the turbulent channel flow using stochastic estimation to approximate the ideal model. In Section 2, the the channel flow and subgrid term are characterized, and two optimal models are described in Section 3 and Section 4. Finally, concluding remarks are provided in Section 5.

2. LES of Turbulent Channel Flow

Optimal LES requires as input the two-point correlations that appear in (6). The correlation data was determined from the direct numerical simulation data of Moser *et al* (1999), at $Re_\tau = 590$.

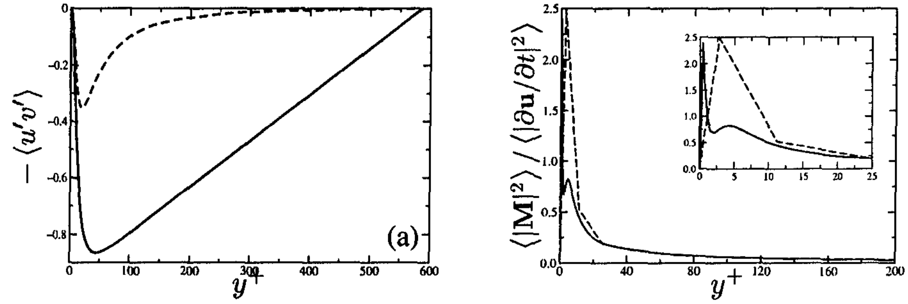


Figure 1. Magnitude of the subgrid force; (a) subgrid contribution to mean Reynolds stress (– – – –) and total Reynolds stress (solid); and (b) mean-square subgrid force fluctuations, normalized by mean-square velocity time derivative fluctuations.

For simplicity we consider a Fourier cut-off filter in the homogeneous directions parallel to the wall. The cut-off wavenumbers in the streamwise and spanwise directions are $k_x h = 16$ and $k_z h = 32$, where h is the channel half-width. This results in an LES representation with 32 modes in the filtered directions, compared to 384 in the DNS. The LES grid spacing is then 116 and 58 wall units in the streamwise and spanwise directions respectively. This is a coarse filter for this flow, which results in a significant model term, as shown in Figure 1. The model terms accounts for as much as 70% of $\widetilde{\partial u_i'/\partial t}$ near the wall and 30% of the mean Reynolds stress. This is representative of the high-Reynolds number wall modeling problem, but with such a large contribution to Reynolds stress, Smagorinsky-based subgrid models must fail (Jiménez and Moser, 1999).

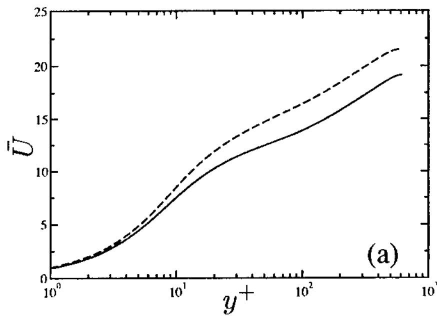
One further complication that arises due to the inhomogeneity in the channel flow is that large quantities of statistical data are required to represent the two-point correlation in the inhomogeneous y direction (Balachandar and Najjar, 2000); more statistical samples than are available from the DNS. To reduce the data required, we only consider estimates that are local in y, though this introduces limitations on the veracity of the resulting models (Volker, 2000).

3. Optimal LES with Directly Estimated Subgrid Terms

Perhaps the most straight-forward way to formulate an optimal LES of the channel flow is to directly estimate the fluctuating subgrid force in terms of the velocity and its y derivatives. In this case the estimate reads

$$m_i(x,y,z) = \langle M_i \rangle + \int_{x,z} L_{ij}(x-x', y, z-z') E_j(x', y, z')\, dx'\, dz' \qquad (7)$$

where the integrals are in x and z only, since the estimates are local in y. This local direct estimate of the subgrid force has been used as a model in an LES. Some representative results are shown in Figure 2. Clearly, the optimal model has not done a particularly good job. The mean velocity is in poor agreement with



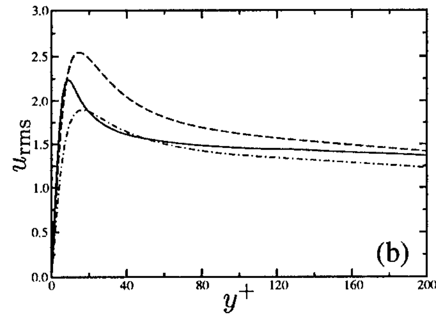


Figure 2. LES results in the channel at $Re_\tau = 590$; (a) mean velocity profiles, and (b) rms streamwise velocity. Shown are the data from the LES (———), DNS (— — —) and filtered DNS (– · – · –).

the DNS, primarily because the wall shear stress (C_f) is over predicted by 11%. The rms streamwise velocity near the wall is also over-predicted compared to the filtered DNS quantity.

The reason for this poor performance can be found by examining the interchange of energy between the resolved and subgrid scales. This is given by $\langle u_i M_i \rangle$, which is plotted in Figure 3. Note that near the wall ($y^+ < 10$), this quantity is positive, indicating energy transfer into the resolved scales. Because of the linearity of the model, this leads to exponential growth of resolved scale fluctuations, until some other effect limits that growth (e.g. nonlinearity). However, this energy transfer term is suggestive of *transport* in the y direction as observed by Häertel & Kleiser (1998). In their analysis, the subgrid energy transfer is rewritten:

$$\langle \tilde{u}_i M_i \rangle = \left\langle \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j} \right\rangle = \frac{\partial \langle \tilde{u}_i \tau_{ij} \rangle}{\partial x_j} - \frac{1}{2} \langle S_{ij} \tau_{ij} \rangle \, , \tag{8}$$

where the first term on the right-hand side represents subgrid transport in y and the second term represents local energy transfer between resolved and subgrid scales (subgrid dissipation for short). These contributions are also shown in Figure 3. Note that the transport term is responsible for the near-wall peak in $\langle \tilde{u}_i M_i \rangle$. The subgrid dissipation term is negative near the wall, though there is a region of true energy transfer to resolved scales farther from the wall.

Since w_i is one of the estimation event variables, the quantity $\langle w_i M_i \rangle$ is predicted exactly by the optimal model, in the *a priori* sense. But, because the estimates are local in y, they do not correctly represent the transport component of this term. To represent the subgrid transport and dissipation terms exactly in the *a prior* sense, we must construct the estimates such that $\langle \tilde{u}_i \tau_{ij} \rangle$ and $\langle S_{ij} \tau_{ij} \rangle$ are recovered (Volker, 2000). Taking advantage of homogeneity in the x and z directions, this can be accomplished by writing M_i as

$$M_i = \rho_i + \frac{\partial \gamma_i}{\partial x_2} \qquad \text{where} \qquad \rho_i = -\frac{\partial \tau_{i1}}{\partial x_1} - \frac{\partial \tau_{i3}}{\partial x_3}, \qquad \gamma_i = -\tau_{i2}, \tag{9}$$

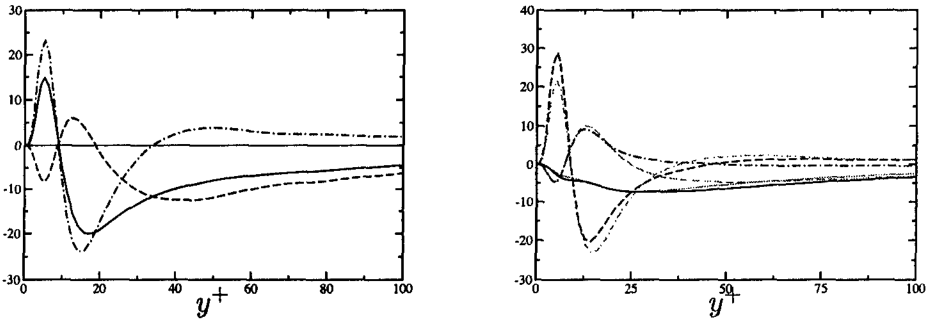


Figure 3. Contributions to the transfer of energy from the resolved scales (left); total energy transfer (———), subgrid dissipation (— — —), subgrid transport (- · - · -). Also, (right) contributions of ρ_i and γ_i to subgrid energy transfer computed *a posteriori* from LES realizations (black), and *a priori* from DNS (grey); $\langle u_i \rho_i \rangle$ (———), $-\langle \gamma_i \partial u_i / \partial x_2 \rangle$ (— — —) $\langle \partial \gamma_i u_i / \partial x_2 \rangle$ (- · - · -)

and estimating ρ_i and γ_i in terms of u_i and $\partial u_i / \partial y$. The total subgrid energy transfer then becomes

$$\langle u_i M_i \rangle = \langle u_i \rho_i \rangle + \left\langle u_i \frac{\partial \gamma_i}{\partial x_2} \right\rangle = \underbrace{\langle u_i \rho_i \rangle - \left\langle \gamma_i \frac{\partial u_i}{\partial x_2} \right\rangle}_{\text{dissipation}} + \underbrace{\left\langle \frac{\partial u_i \gamma_i}{\partial x_2} \right\rangle}_{\text{transport}} \tag{10}$$

where each of the terms on the right hand side are represented exactly *a priori*.

The *a priori* contributions of the individual terms in (10) to the total subgrid energy transfer are shown in Figure 3. Note that it is the dissipation term involving γ that is responsible for the augmentation of energy in the resolved scales. Because of the structure of this term, it will not produce unchecked exponential growth, as occurs when M is estimated directly.

4. Optimal LES with Subgrid Transport Estimated

An LES based on the local estimation of ρ and γ with event data consisting of the velocities and their y derivatives was performed. The results are shown in Figure 4 for mean and rms velocities. The difficulties with the mean velocity, and over-prediction of C_f are now gone. For the rms velocities, the LES are now much closer to the values from the filtered DNS, though there are still some minor discrepancies. A variety of other statistical quantities, including spectra and two-point correlations are in reasonably good agreement with those of the filtered DNS (Volker, 2000).

Given the apparent importance of the energy transfer and transport terms, the *a posteriori* prediction of these quantities in the LES are of some importance. They are shown in black in Figure 3. The agreement with the *a priori* results (grey) is reasonably good, suggesting that ensuring that a model is *a priori* accurate of for these quantities will lead to reasonable predictions.

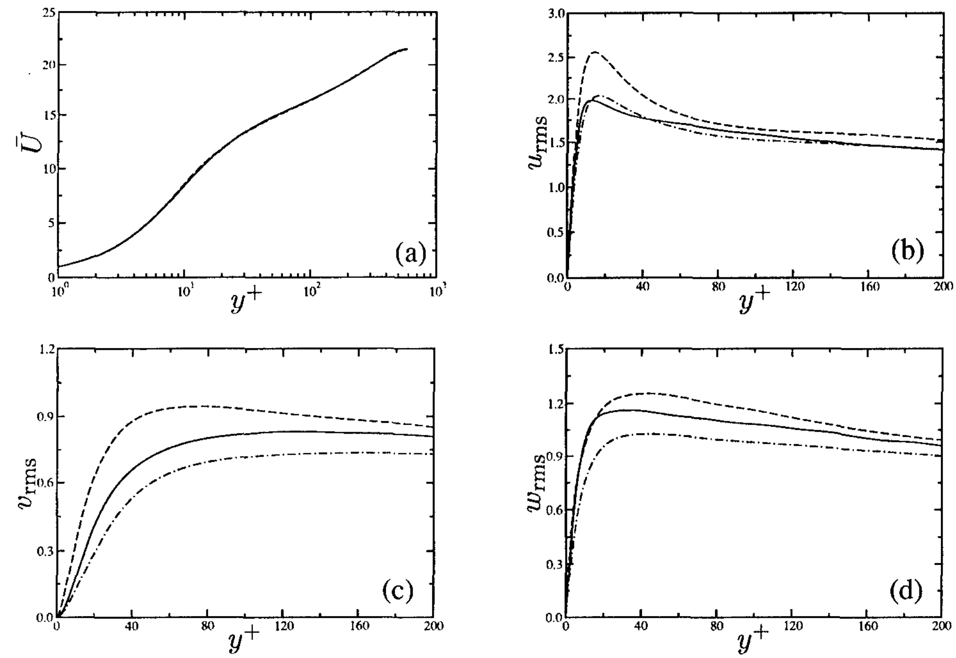


Figure 4. Channel LES using optimal estimation of ρ and γ (see (9)); (a) mean velocity profile, (b) rms streamwise velocity fluctuations, $u_{\rm rms}$, (c) rms wall-normal velocity fluctuations, $v_{\rm rms}$, and (d) rms spanwise velocity fluctuations, $w_{\rm rms}$. Shown are data from the LES (———), the DNS (— — —) and the filtered DNS (- · - · -).

5. Discussion and Conclusions

In both isotropic turbulence (Langford, 2000) and the channel flow, a properly constructed optimal LES model produces very good simulations. But, the experience with the channel flow suggests that what makes a "properly constructed" optimal model is not necessarily obvious. What seems to be important is that the optimal models *a priori* reproduce essential statistical properties of the filtered turbulence, such as the "dissipation" of resolved scale energy (i.e. transfer to subgrid scales) and the subgrid contribution to energy transport. The optimal models actually *a priori* represent more than just these energy dynamic terms. For example in isotropic turbulence, they *a priori* represent the dynamics of the two-point correlation tensor (Langford and Moser, 1999).

In the channel, a global linear optimal model would also reproduce all subgrid contributions to the two-point correlation dynamics. However, we were not able to construct a global estimate, because of the limited statistical data available. The estimates we built, were designed to represent the energy transport term, as well as the dissipation, but they also reproduce the subgrid contribution to transport and dissipation terms in the resolved-scale Reynolds stress transport equations. Only the velocity-pressure-gradient term in the Reynolds stress equations is not reproduced *a priori*. We speculate that a model that also reproduces this term would further improve LES performance, and that a global model that reproduces 2-point correlation dynamics would do even better.

The optimal models developed here rely on extensive correlation data that we obtained from DNS. Clearly, these models would not be practical if such detailed data were required for each flow in which one wishes to perform an LES. It is thus necessary to generalize models such as those devised here to be applicable in a broad range of applications.

Acknowledgements

The research reported here has been supported by NSF and AFOSR through grant CTS-9616219, NSF grant CTS-001435 and AFOSR grant F49620-01-1-0181, and by the Center for Simulation of Advanced Rockets, which is funded by the Department of Energy through University of California grant B341494. This support is gratefully acknowledged.

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